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A statistical mechanics analysis of the set covering problem

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Abstract. The dependence of the optimal solution average cost E_{m} of the set covering problem on the density of 1's of the incidence matrix (β) and on the number of constraints (P) is investigated in the limit where the number of items (N) goes to infinity. The annealed approximation is employed to study two stochastic models: the constant density model, where the elements of the incidence matrix are statistically independent random variables, and the Karp model, where the rows of the incidence matrix possess the same number of 1's. Lower bounds for E_{m} are presented in the case that P scales with $\ln N$ and β is of order 1, as well as in the case that P scales linearly with N and β is of order $1/N$. It is shown that in the case that P scales with $\exp N$ and β is of order 1 the annealed approximation yields exact results for both models.

1. Introduction

The set covering problem (SCP) is the zero–one integer program which consists of finding the N-tuples $\boldsymbol{s} = (s_1, s_2, \ldots, s_N)$ that minimize the cost function

$$E = \sum_i^N c_i s_i \tag{1.1}$$

subject to P linear constraints

$$\sum_i^N a_{ki} s_i \geqslant e_k \qquad k = 1, \ldots, P \tag{1.2}$$

and to the integrality constraint $s_i = 0, 1$. Here $a_{ki} = 0, 1$ are the elements of the P by N incidence matrix $\mathcal{A}$, c_i is the cost assigned to element i, and $e_k = 1\,\forall k$. In this paper we consider the so-called minimum cardinality set covering problem for which $c_i = 1\,\forall i$. A particular instance of the SCP is then solely determined by the matrix $\mathcal{A}$. If we think of the columns of $\mathcal{A}$ and $\boldsymbol{e}$ as sets, then this problem is equivalent to finding the cheapest union of sets from $\mathcal{A}$ that covers every component of $\boldsymbol{e}$, hence the denomination set covering for this combinatorial optimization problem.

Despite its simplicity, the SCP models numerous practical situations such as airline crew scheduling, political redistricting and information retrieval (Salkin 1975), to name only the most classical ones. Moreover, the SCP was shown to belong to the NP-complete class (Karp 1972), which basically means that there is no known deterministic algorithm guaranteed to solve all instances of this problem within a polynomial time bound (Garey and Johnson 1979). However, since the NP-completeness proof is a worst-case analysis, it does not tell us anything about particular instances of the SCP, which actually may be solvable by a

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polynomially bound heuristic algorithm. There is an alternative and perhaps complementary method of analysis, so-called average-case analysis, where, instead of considering the worst possible instance, the analyses focus on a set of 'typical' instances. The difficulty, of course, is to generate a collection of typical instances for problems we encounter in practice. Usually, a stochastic model is assumed for the matrix $\mathcal{A}$, which functions then as a generator of random instances. The obvious drawback of this approach is that we can seldom know what probability distribution for the elements of $\mathcal{A}$ is realistic. Nevertheless, there is a great practical interest in the analysis of the stochastic models since they are widely employed in the evaluation of the performance of heuristic algorithms. We note that for the stochastic models one expects that the larger the size of the problem N, the more unlikely the worst cases are to arise.

The relevant parameter to describe a stochastic model for the minimum cardinality set covering problem is the density of 1's of the matrix $\mathcal{A}$, because the feasibility of an instance depends on the existence of at least one 1 in each row of $\mathcal{A}$. In this note we study two stochastic models, which we describe in the following. The first model, due to Gimpel (1967), is termed the *constant density model* because it assumes that the elements a_{ki} are statistically independent random variables drawn from the distribution

$$\mathcal{P}_{cd}(\{a_{ki}\}) = \prod_{ik} \beta\delta(a_{ki} - 1) + (1 - \beta)\delta(a_{ki}) \tag{1.3}$$

where $0 \leqslant \beta \leqslant 1$ is a control parameter measuring the density of 1's of the incidence matrix, and $\delta(x)$ is the Dirac delta function. This model was studied by Vercellis (1984) in the case when $N \to \infty$ and $P = \exp(\alpha N)$, where α is a parameter independent of N. The main result of Vercellis' analysis is that the optimal cost is given by

$$E_{\mathrm{m}} = -\frac{\ln P}{\ln(1 - \beta)} \tag{1.4}$$

with a probability of 1. Actually, the method employed by Vercellis to derive this result was to show that the expression in the RHS of equation (1.4) is both an upper bound and a lower bound for E_{m}.

The second model, proposed by Karp (1976), assumes that there is a constant number of 1's in each row of the incidence matrix, i.e.

$$\sum_{i}^{N} a_{ki} = b \qquad k = 1, \ldots, P. \tag{1.5}$$

Thus the elements a_{ki} $i = 1, \ldots, N$ are no longer statistically independent random variables. Elements belonging to different rows of $\mathcal{A}$, however, are statistically independent. The ratio $\beta = b/N$ gives then the density of 1's of the incidence matrix. The probability distribution we use to describe this model, which we will refer to as the *Karp model*, is the following

$$\mathcal{P}_{k}(\{a_{ki}\}) = \prod_{k} \frac{1}{C(N, k)} \delta\left(b, \sum_{i} a_{ki}\right) \tag{1.6}$$

where $\delta(k, l)$ is the Kronecker delta and we have introduced the notation $C(n, k) = n!/k!(n - k)!$. To the best of our knowledge, the dependence of the optimal cost E_{m} on the parameters β and P is not known for this model.

Statistical mechanics techniques developed in the study of spin glasses, namely, the replica method and the annealed approximation, have been applied successfully in the study of stochastic models of several classical combinatorial optimization problems, such as the graph partitioning problem (Fu and Anderson 1986), the matching problem (Mèzard and Parisi 1985), and the travelling salesman problem (Vannimenus and Mèzard 1984, Mèzard

and Parisi 1986). We refer the reader to Mèzard *et al* (1987) for a thorough presentation of the accomplishments of the statistical mechanics approach to the analysis of optimization problems. In general, these analyses are restricted to the calculation of the average value of the optimal cost solutions. More recently, Barbato and Fontanari (1994) have employed the statistical mechanics approach to obtain microscopic information about the structure of the minimum weight solutions to a set of linear equations. It should be mentioned that other techniques from statistical physics, besides those mentioned above, can also be useful to the analysis of combinatorial problems. A remarkable example of this is the recent application of finite-size scaling to study threshold phenomena in the k-satisfiability problem (Kirkpatrick and Selman 1994).

In this paper we employ the annealed approximation in the microcanonical ensemble formalism of the statistical mechanics (Fontanari and Meir 1993) to derive rigorous lower bounds to E_{m} as a function of β and P for both stochastic models described above. The basic quantity we must evaluate in this formalism is $\mathcal{N}(E)$, the number of N-tuples $\boldsymbol{s}$ that possess cost E and satisfy constraints (1.2). It is given by

$$\mathcal{N}(E) = \sum_{\{\boldsymbol{s}\}} \delta\Big(E, \sum_i s_i\Big) \prod_k \Theta\Big(\sum_i a_{ki} s_i - 1\Big) \tag{1.7}$$

where the summation is over the 2^N binary N-tuples $\boldsymbol{s}$, and $\Theta(x) = 1$ if $x \geqslant 0$ and 0 otherwise. The standard procedure to obtain physically meaningful results is to take averages over extensive quantities only (Binder and Young 1986). In this case, the extensive quantity associated to $\mathcal{N}$ is the average entropy $\mathcal{S}(E) = \langle \ln \mathcal{N}(E) \rangle$, where $\langle\ \rangle$ stands for the averages over the random variables a_{ki}. The optimal profit E_{m} is determined by the conditions $\mathcal{S}(E < E_{\mathrm{m}}) \to -\infty$ and $\mathcal{S}(E \geqslant E_{\mathrm{m}}) \geqslant 0$. The annealed approximation consists of taking the averages within the logarithm function, i.e.

$$\mathcal{S}_{\mathrm{a}}(E) = \ln \langle \mathcal{N}(E) \rangle. \tag{1.8}$$

Note that while $\mathcal{S}$ is clearly non-negative or $-\infty$, the annealed entropy $\mathcal{S}_{\mathrm{a}}$ can take on finite negative values. Although the annealed entropy is physically meaningless, it gives a rigorous upper bound to the correct entropy, i.e. $\mathcal{S}_{\mathrm{a}} > \mathcal{S}$, which can easily be demonstrated using the convexity of the logarithm function. In particular, this inequality implies that $\mathcal{S} \to -\infty$ for any E such that $\mathcal{S}_{\mathrm{a}}(E) < 0$. This is the reason why the value of E at which $\mathcal{S}_{\mathrm{a}}$ vanishes, E_{m}^{a}, is a rigorous lower bound to E_{m} (Fontanari and Meir 1993). A similar technique for deriving lower bounds was employed by Vannimenus and Mèzard (1984) using the canonical ensemble formalism. In fact, as pointed out by those authors, the annealed approximation is very useful to determine the appropriate scaling of P with N in the asymptotic limit $N \to \infty$. In particular, for the set covering problem we have found that only two scalings give non-trivial instances for β independent of N, namely, $P = \alpha \ln N$ and $P = \exp(\alpha N)$. By a trivial instance we mean either an unfeasible instance ($\mathcal{S}_{\mathrm{a}} < 0\, \forall E$) or an easy instance ($\mathcal{S}_{\mathrm{a}} > 0\, \forall E$).

Besides the calculation of the annealed entropy, which demands the evaluation of the first moment of the distribution of $\mathcal{N}$, we also calculate its variance $\sigma^2 = \langle \mathcal{N}^2 \rangle - \langle \mathcal{N} \rangle^2$. The vanishing of this quantity implies that $\mathcal{N}$ is a self-averaging quantity and hence that the annealed approximation yields exact results. In fact, we show that σ^2 tends to zero in the regime studied by Vercellis (1984) and, as expected, that the results of the annealed approximation coincide with the results obtained by Vercellis for the constant density model. Our approach, however, can also be applied to the Karp model, in contrast to Vercellis' which is limited to the constant density model.

The remainder of this paper is organized as follows. In section 2 we study the constant density model and in section 3 we consider the Karp model. In the appendix we employ the replica method to derive the linear programming relaxation lower bound for the constant density model in the regime where E is of order 1. Finally, in section 4 we discuss our results and present some concluding remarks.

2. The constant density model

We proceed now with the explicit calculation of the annealed entropy in the case when the elements a_{ki} are distributed according to equation (1.3). Using the integral representations for the theta and the Kronecker delta functions, it is straightforward to carry out the averages over a_{ki} and the summation over s. The final result, valid for all values of N and P, is simply

$$\mathcal{S}_{\mathrm{a}}(E) = \ln C(N, E) + P \ln[1 - (1 - \beta)^E]. \tag{2.1}$$

In the following we will focus on the asymptotic limit $N \to \infty$ with β independent of N. One the one hand, if E is of order 1 then $\ln C(N, E)$ will be of order $\ln N$ and, therefore, a non-trivial regime can be achieved only by the scaling $P = \alpha \ln N$. On the other hand, if E is of order N, say $E = \epsilon N$, then $\ln C(N, E)$ will also be of order N, while the second term of the RHS of equation (2.1) will be of order $P \exp(-N)$. Thus, we can find a non-trivial regime only if P scales with $\exp(\alpha N)$.

We consider first the scaling $P = \alpha \ln N$. In this case, neither the cost E nor the annealed entropy $\mathcal{S}_{\mathrm{a}}$ are extensive quantities. In fact we have

$$\mathcal{S}_{\mathrm{a}}(E)/\ln N = E + \alpha \ln[1 - (1 - \beta)^E]. \tag{2.2}$$

As mentioned in the introduction, the solution $E_{\mathrm{m}}^{\mathrm{a}}$ of the equation $\mathcal{S}_{\mathrm{a}}(E) = 0$ is a lower bound for the minimum cost E_{m}. This lower bound is presented in figure 1 as a function of β for several values of α. In this regime, all instances of the SCP seem to be feasible, since given α and β it is always possible to find an arbitrarily large E for which $\mathcal{S}_{\mathrm{a}}$ is positive. We must mention that within the annealed approximation framework we cannot guarantee that an instance is feasible; we can, however, guarantee that it is unfeasible, since $\mathcal{S}_{\mathrm{a}} < 0$ implies $\mathcal{S} \to -\infty$.

The simplest method to obtain lower bounds to integer programming problems is probably the linear programming (LP) relaxation, which consists of relaxing the integrality constraint on the variables s_i, so that $0 \leqslant s_i \leqslant 1$. The exact analytical derivation of this bound, presented in the appendix, is possible only in the case when E is of order 1, which implies that s_i is of order $1/N$. The final result, which does not depend on P, is simply

$$E_{\mathrm{m}}^{\mathrm{LP}} = \frac{1}{\beta}. \tag{2.3}$$

From the comparison between the LP relaxation bound and the annealed one, presented in figure 1, we conclude that the latter is tighter than the former for large P only.

In the regime considered by Vercellis (1984), where $P = \exp(\alpha N)$ and $E = \epsilon N$, the annealed entropy becomes

$$\mathcal{S}_{\mathrm{a}}(\epsilon)/N = -\epsilon \ln \epsilon - (1 - \epsilon) \ln(1 - \epsilon) - \frac{1}{N} \exp\{N[\alpha + \epsilon \ln(1 - \beta)]\}. \tag{2.4}$$

Thus the value of the annealed entropy is determined by the sign of the argument of the exponential. More specifically, if $\epsilon > \epsilon_{\mathrm{m}}^{\mathrm{a}}$, where

$$\epsilon_{\mathrm{m}}^{\mathrm{a}} = -\frac{\alpha}{\ln(1 - \beta)} \tag{2.5}$$

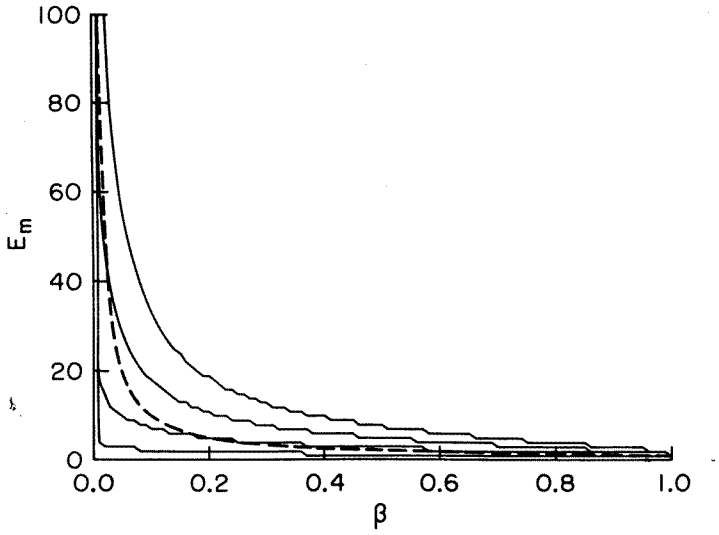

Fig. 1

Figure 1. Annealed lower bound for the optimal cost as a function of the density of 1's β of the incidence matrix for, from bottom to top, $\alpha = P/\ln N = 1, 10, 100$ and 1000. The constant density model and the Karp model give identical results in this limit. The broken curve is the linear programming relaxation lower bound for the constant density model.

then $\mathcal{S}_a/N = -\epsilon \ln \epsilon - (1-\epsilon)\ln(1-\epsilon)$, while $\mathcal{S}_a \to -\infty$, otherwise. Since $\alpha = \ln P/N$, the annealed lower bound $E_m^a = \epsilon_m^a N$ is identical to the optimal cost derived by Vercellis (1984). Furthermore, the annealed entropy is positive or $-\infty$, presenting then a physically correct behaviour. These results lead us to conjecture that the annealed calculation may be exact for this scaling. To prove this we calculate the variance $\sigma^2 = \langle \mathcal{N}^2 \rangle - \langle \mathcal{N} \rangle^2$ of the distribution of $\mathcal{N}$. Following a procedure analogous to that used for the calculation of $\langle \mathcal{N} \rangle$ we find

$$\sigma^2 = \sum_{n=0}^{E} \frac{N!}{n!(N-2E+n)![(E-n)!]^2} [\Gamma_n^P - \Lambda^{2P}] \tag{2.6}$$

where

$$\Gamma_n = 1 - 2(1-\beta)^E + (1-\beta)^{2E-n} \tag{2.7}$$

and

$$\Lambda = 1 - (1-\beta)^E. \tag{2.8}$$

Similarly to equation (2.1), this equation is exact for all values of N and P. The following discussion, however, will be restricted to the case $P = \exp(\alpha N)$ and $N \to \infty$. Moreover, since our goal is to prove the exactness of the annealed results we consider only the regime $\epsilon > \epsilon_m^a$ for which $\mathcal{S}_a/N$ is finite. In this case we have

$$\alpha < -\epsilon \ln(1-\beta) \tag{2.9}$$

so that $\Lambda^{2P} \to 1$. Thus, to show that σ^2 vanishes we must show that $\Gamma_n^P \to 1$ for all n. However, this is obvious because if α satisfies the inequality (2.9) then it will also satisfy $\alpha < -(2\epsilon - n/N)\ln(1-\beta) \forall n$, that implies $\Gamma_n^P \to 1$.

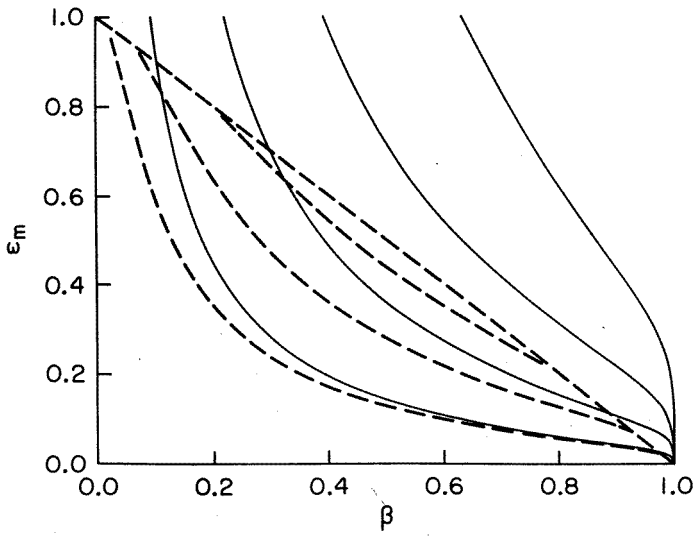

Fig. 2

Figure 2. Exact optimal cost for the constant density model (full curves) and for the Karp model (broken curves) as functions of the density of 1's β of the incidence matrix for, from bottom to top, $\alpha = \ln P/N = 0.1, 0.25, 0.5$ and 1.

In figure 2 we present $\epsilon_{\rm m}^{\rm a} = \epsilon_{\rm m}$ as a function of β for several values of α. We note that in the regime $\alpha > -\ln(1-\beta)$ the instances are unfeasible, i.e. the constraints cannot be satisfied even if $\epsilon = 1$.

Finally, we consider the linear scaling $P = \alpha N$. This scaling yields non-trivial instances provided that E is extensive and β vanishes like N^{-1}, say $\beta = \hat{\beta}/N$ with $\hat{\beta}$ of order 1. In this case we have

$$\mathcal{S}_{\rm a}(\epsilon)/N = -\epsilon \ln \epsilon - (1-\epsilon)\ln(1-\epsilon) + \alpha \ln[1 - \exp(-\epsilon\hat{\beta})]. \tag{2.10}$$

The annealed lower bound $\epsilon_{\rm m}^{\rm a}$ is presented in figure 3 as a function of $0 < \hat{\beta} < \infty$ for several values of α. Similar to the case discussed before, there are regions in the plane $(\hat{\beta}, \alpha)$ where the instances are unfeasible; the curves for $\epsilon_{\rm m}^{\rm a}$, however, do not begin from $\epsilon_{\rm m}^{\rm a} = 1$.

3. The Karp model

In this section we consider the stochastic model proposed by Karp (1976) in which the rows of the incidence matrix $\mathcal{A}$ possess the same number of 1's. Although the elements a_{ki} belonging to the same row of $\mathcal{A}$ are no longer statistical independent variables, the calculations are no more involved than for the constant density model. In fact, using the integral representations for the Kronecker deltas and for the theta function we can easily obtain

$$\mathcal{S}_{\rm a}(E) = \ln C(N, E) + P \ln\left(1 - \frac{C(N-E, b)}{C(N, b)}\right) \tag{3.1}$$

where we have used the distribution (1.6) to carry out the average over the elements a_{ki}. This equation is valid for all values of N and P. Note the correspondence between the

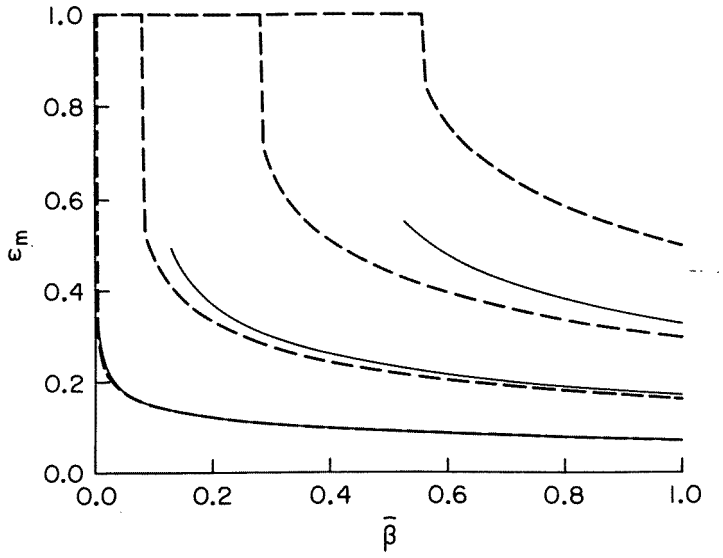

Fig. 3

Figure 3. Annealed lower bound for the optimal cost in the limit where the density of 1's β vanishes like $\hat{\beta}/N$ for, from bottom to top, $\alpha = P/N = 0.1, 0.25, 0.5$ and 1. The curve for $\alpha = 1$ for the constant density model does not appear in the figure because the problem is unfeasible in the range of $\hat{\beta}$ considered. The convention is the same as in figure 2.

ratio $\gamma = C(N-E,b)/C(N,b)$ and the term $(1-\beta)^E$ of the analogous equation, equation (2.1), for the constant density model.

In the following we shall focus on the possible scalings of P with N when $N \to \infty$. Let us consider the case where b is extensive, say $b = \beta N$. If E is of order 1 then $\gamma \to (1-\beta)^E$ and, therefore, this model becomes equivalent to the constant density model. The behaviour of the lower bound $E_{\mathrm{m}}^{\mathrm{a}}$ is presented in figure 1. Similarly to the constant density model, in the case where $E = \epsilon N$ a non-trivial regime occurs only if $P = \exp(\alpha N)$, yielding the following annealed entropy:

$$\mathcal{S}_{\mathrm{a}}(\epsilon)/N = -\epsilon \ln \epsilon - (1-\epsilon)\ln(1-\epsilon) - \frac{1}{N}\Theta(1-\epsilon-\beta)\exp(N\Xi) \qquad (3.2)$$

where

$$\Xi = \alpha + (1-\epsilon)\ln(1-\epsilon) + (1-\beta)\ln(1-\beta) - (1-\epsilon-\beta)\ln(1-\epsilon-\beta). \qquad (3.3)$$

Hence $\mathcal{S}_{\mathrm{a}} \to -\infty$ if $\Xi > 0$ and $\mathcal{S}_{\mathrm{a}}$ is positive and finite otherwise. Thus the annealed estimate $\epsilon_{\mathrm{m}}^{\mathrm{a}}$ for the minimal cost is obtained by solving $\Xi(\epsilon) = 0$ for α and β fixed. As for the constant density model, we can prove that the annealed lower bound is exact by showing that the variance $\sigma^2 = \langle \mathcal{N}^2 \rangle - \langle \mathcal{N} \rangle^2$ vanishes in this limit. More specifically, σ^2 is still given by equation (2.6), except that Γ_n and Λ are now given by

$$\Gamma_n = 1 - 2\frac{C(N-E,b)}{C(N,b)} + \frac{C(N-2E+n,b)}{C(N,b)} \qquad (3.4)$$

and

$$\Lambda = 1 - \frac{C(N-E,b)}{C(N,b)}. \qquad (3.5)$$

In the regime $\epsilon > \epsilon_{\rm m}^{\rm a}$ we can easily show that $\Gamma_n^P \to \Lambda^{2P} \to 1\,\forall n$ so that $\sigma^2 \to 0$. The exact minimal cost density $\epsilon_{\rm m}^{\rm a} = \epsilon_{\rm m}$ is presented in figure 2 together with the results for the constant density model. Note that the instances of the SCP generated by the Karp model are always feasible for $\beta > 0$.

Similarly to the constant density model, the linear scaling $P = \alpha N$ gives non-trivial results if b is of order 1. Actually, this corresponds to a vanishing density of 1's, $\beta = b/N \to 0$, so b plays the same role as the parameter $\hat{\beta}$ introduced in the analysis of the constant density model. In this case, the annealed entropy equation (3.1) becomes

$$\mathcal{S}_{\rm a}(\epsilon)/N = -\epsilon \ln \epsilon - (1-\epsilon)\ln(1-\epsilon) + \alpha \ln[1-(1-\epsilon)^b]. \tag{3.6}$$

The lower bound $\epsilon_{\rm m}^{\rm a}$ is presented in figure 3 together with the results for the constant density model. The main difference between these models is that the Karp model always generates feasible instances. We note that the two models yield very similar results for small α or large $\hat{\beta}$. In fact, this occurs wherever $\epsilon_{\rm m}^{\rm a}$ is small.

4. Conclusion

Despite the importance of the set covering problem, attested by the numerous practical situations it models (Salkin 1975), there are very few average-case results concerning the dependence of the optimal cost $E_{\rm m}$ on the parameters that describe the stochastic models. Typically, these parameters are the density of 1's of the incidence matrix (β), the number of constraints (P), and the number of items (N). In general, the analyses are restricted to the evaluation of the performance of greedy heuristic algorithms in test problems generated by the stochastic models (Gimpel 1967, Karp 1976). The only exception we know is the work of Vercellis (1984) which, although explicitly focusing on the dependence of $E_{\rm m}$ on the control parameters, is limited to a particular scaling of P with N and to a specific stochastic model, namely, the constant density model. In this sense, we believe that the results presented in this paper which extend the results of Vercellis (1984) to the Karp model and give lower bounds for the optimal cost for different scalings of P with N, are relevant contributions to the statistical analysis of the SCP. As mentioned before, the usefulness of this type of analysis is due to the extensive use of stochastic models to generate test problems for the evaluation of the performance of heuristic algorithms.

Besides the annealed approximation and the linear programming relaxation discussed in this paper, another powerful method for generating lower bounds for integer programming problems is the Lagrangian relaxation technique (Beasley 1993). For the set covering problem, it consists of relaxing the constraints (1.2) by introducing P Lagrangian multipliers $\lambda = (\lambda_1, \ldots, \lambda_P)$ into the cost function (1.1) which is then written as

$$L(\lambda, \boldsymbol{s}) = \sum_k^P e_k \lambda_k + \sum_i^N s_i \left(1 - \sum_k \lambda_k a_{ki}\right) \tag{4.1}$$

with $\lambda_k > 0$. For λ fixed, the N-tuple $\boldsymbol{s}$ that minimizes L is simply $s_i = 1$ if $\sum_k \lambda_k a_{ki} \geqslant 1$, and $s_i = 0$ otherwise. Hence

$$L(\lambda) = \sum_k^P e_k \lambda_k - \sum_i^N \left(\sum_k \lambda_k a_{ki} - 1\right)\Theta\left(\sum_k \lambda_k a_{ki} - 1\right). \tag{4.2}$$

It can be easily shown that $L(\lambda)$ gives a lower bound for $E_{\rm m}$ for any λ (Beasley 1993). The problem then becomes to find the set of Lagrangian multipliers that maximizes L. This procedure has been applied successfully to a similar integer programming problem, namely, the knapsack problem (Meanti *et al* 1990, Fontanari 1995). In the knapsack problem, the a_{ki}

are real-valued random variables uniformly distributed over the unit interval and $e_k = eN$ with e of order 1. In fact, it can be shown that the Lagrangian relaxation bound converges to the optimal cost solution with probability 1 in the regime where N goes to infinity and P remains finite (Meanti *et al* 1990). Basically, the reason why this approach does not work for the set covering problem is because $e_k = 1$ is non-extensive and therefore, for N large, only the second term in the RHS of equation (4.2) will contribute to L. Since this term is negative or zero, the maximum value that L can take on is $L = 0$, which is a quite trivial lower bound.

Finally, we should mention that the difficulty in applying the replica formalism to the set covering problem or to the knapsack problem, that would allow for the exact calculation of the average entropy, is the basic requirement that the elements a_{ki} must be positive. As a result, the average over a_{ki} generates effective high-order interactions between the replicas, which makes the analysis extremely involved. Thus, it seems that the annealed approximation is the only technique that can be used to study the SCP in the diverse non-trivial regimes of scaling of P with N.

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Appendix

In this appendix we calculate the linear programming (LP) relaxation lower bound for the optimal cost of the minimum cardinality set covering problem for the constant density model in the regime where E is of order 1. In the LP relaxation the problem is again to minimize the cost (1.1) subject to constraints (1.2), except that now the s_i's are real-valued variables, i.e. $0 \leqslant s_i \leqslant 1$. Since the entropy can take on negative values in this case, we employ the canonical ensemble formulation of the statistical mechanics to obtain the LP relaxation lower bound. It is given by

$$E_{\mathrm{m}}^{\mathrm{LP}} = -\lim_{h\to\infty} \frac{\partial}{\partial h} \langle \ln Z(h) \rangle \tag{A.1}$$

where

$$Z(h) = \prod_j \int_0^1 \mathrm{d}s_j \exp\left(-h \sum_j s_j \right) \prod_k \Theta\left(\sum_j a_{kj} s_j - 1 \right) \tag{A.2}$$

is the partition function, and h is the inverse of the temperature. The quenched average $\langle\ \rangle$ over the variables a_{kj} is carried out through the replica method. Using the identity

$$\langle \ln Z \rangle = \lim_{n\to 0} \frac{1}{n} \ln \langle Z^n \rangle \tag{A.3}$$

we first evaluate $\langle Z^n \rangle$ for *integer* n and then analytically continue to $n = 0$. The calculation is simple in this limit because if $E = \sum_j s_j$ is of order 1 then s_j must be of order $1/N$, so that we can make expansions in powers of s_j and retain only the leading term. More specifically, using the integral representation for the theta function we are faced with the evaluation of the following average:

$$\mathcal{M} = \left\langle \exp\left(\mathrm{i} \sum_k^P \sum_a^n \sum_j^N \hat{y}_{ka} a_{kj} s_j^a \right) \right\rangle = \prod_{jk} \left\langle \exp\left(\mathrm{i} a_{kj} \sum_a^n \hat{y}_{ka} s_j^a \right) \right\rangle \tag{A.4}$$

where $\hat{y}_{ka}$ is a Lagrangian multiplier that appears in the integral representation for the theta function. Note that we have made explicit use of the statistical independence of the a_{kj} that characterize the constant density model. Expanding in powers of s_j and keeping terms of order $1/N$ yields

$$\mathcal{M} \approx \exp\left(\mathrm{i}\sum_{k}^{P}\sum_{a}^{n}\sum_{j}^{N}\hat{y}_{ka}\langle a_{kj}\rangle s_j^a\right) = \exp\left(\mathrm{i}\beta\sum_{k}^{P}\sum_{a}^{n}\sum_{j}^{N}\hat{y}_{ka}s_j^a\right). \tag{A.5}$$

Thus the net result of the average was to replace a_{kj} by its mean β. Introducing the auxiliary variable $E_a = \sum_j s_j^a$, it is straightforward to carry out the integration over s_j^a. The final result is

$$(\langle Z^n\rangle)^{1/n} \approx \int_{-\infty}^{\infty} \mathrm{d}E\,\Theta(\beta E - 1)\,\mathrm{e}^{-hE}\int_{-\infty}^{\infty}\frac{\mathrm{d}\hat{E}}{\pi}\,\mathrm{e}^{\mathrm{i}\hat{E}(2E-N)}\left(\frac{\sin\hat{E}}{\hat{E}}\right)^N. \tag{A.6}$$

Here $\hat{E}$ is the Lagrangian multiplier associated with the auxiliary variable E. Since the averaging procedure has not resulted in an effective interaction between replicas, the n different replicas give the same contribution to the nth moment of Z. Furthermore, due to the replacement of a_{kj} by β the constraints become independent of the index $k = 1, \ldots, P$, and since $\Theta^P(x) = \Theta(x)$ the parameter P disappears altogether from our results. Using the following approximation

$$\left(\frac{\sin x}{x}\right)^N \approx \exp(-Nx^2/6) \tag{A.7}$$

valid for large N, to evaluate the integral over $\hat{E}$ we find

$$\langle \ln Z\rangle \approx -\tfrac{3}{2}N + \tfrac{1}{24}N(h-6)^2 + \ln H\left[\frac{1/\beta + N(h-6)/12}{\sqrt{N/12}}\right] \tag{A.8}$$

where $H(x) = \int_x^\infty \mathrm{d}t/\sqrt{2\pi}\,\mathrm{e}^{-t^2/2}$. Taking the limit $N \to \infty$ yields

$$E^{\mathrm{LP}} = \frac{1}{\beta} + \frac{1}{h-6} \tag{A.9}$$

if $h \geqslant 6$, and

$$E^{\mathrm{LP}} = \tfrac{1}{12}N(6-h) \tag{A.10}$$

otherwise. Here $E^{\mathrm{LP}} = -\partial\langle \ln Z(h)\rangle/\partial h$. Thus our theory is not valid in the regime $h < 6$ since we have assumed that E^{LP} is of order 1. Finally, taking the limit $h \to \infty$ in equation (A.9) yields

$$E_{\mathrm{m}}^{\mathrm{LP}} = \frac{1}{\beta}. \tag{A.11}$$

We emphasize that this result was derived under the assumption of statistical independence of the elements a_{ki} $i = 1, \ldots, N$, which is not valid for the Karp model. We must be careful not to undervalue equation (A.11). Although it is obvious that if the elements a_{ki} are replaced by their means β in (1.2) then the optimal cost must equal $1/\beta$, the calculations presented above show how this replacement can be justified under the assumptions that E is of order 1 and that the elements a_{ki} are statistically independent random variables.

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